AMS210.01.
Homework 2

Andrei Antonenko

Due at the beginning of the class, March 3, 2003

1. Find the following expressions if they are defined, otherwise, specify why they are not defined.

   (a) $\begin{pmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & -1 \end{pmatrix} \begin{pmatrix} 3 & 29 \\ 2 & 18 \\ 0 & -3 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 5 & 3 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 & 5 \\ -1 & 4 & -2 \\ 3 & -1 & 1 \end{pmatrix}$

   (d) $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

   (e) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

   (f) $\begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 3 \\ 2 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & 2 \\ -2 & -1 & 1 & 2 \\ 2 & 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -4 & 6 & 1 \\ 2 & 2 & -5 & -2 \\ 2 & -2 & 6 & 4 \\ 1 & 3 & 0 & 1 \end{pmatrix}$

2. Find the product of matrices $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, where m and n are real numbers.

3. Find the following powers of matrices:

(a) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^2, \quad \alpha \in \mathbb{R}$

(b) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^n, \quad \alpha \in \mathbb{R}, \quad n \in \mathbb{N}$

(c) $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^2, \quad \lambda \in \mathbb{R}$

(d) $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^3, \quad \lambda \in \mathbb{R}$

(e) $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n, \quad \lambda \in \mathbb{R}, \quad n \in \mathbb{N}$

4. Find $f(A)$ if

(a) $f(x) = x^2 + x + 1; \qquad A = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$

(b) $f(x) = x^3 - 3x + 2; \qquad A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

5. For all values of $n \in \mathbb{N}$ find

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^n$$

6. Recall, that by I_{ij} we denote the matrix which has 1 on (i, j)-th place and zeros on all other places. For any matrix A compute AI_{ij} and $I_{ij}A$.

7. Solve the following systems of matrix equations:

(a) $\begin{cases} X + Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ 2X + 3Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{cases}$

(b) $\begin{cases} X + 2Y = \begin{pmatrix} 9 & 8 \\ 7 & 6 \end{pmatrix} \\ X - Y = \begin{pmatrix} -3 & -1 \\ 1 & 3 \end{pmatrix} \end{cases}$

8. Solve the following matrix equations:

(a) $\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

(b) $X \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 6 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix} X = \begin{pmatrix} 1 & -3 & 0 \\ 10 & 2 & 7 \\ 10 & 7 & 8 \end{pmatrix}$

9. Find the inverses of the following matrices:

(a) $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$

(b) $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 3 & 0 \\ 2 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

(e) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}$

(f) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

(g) $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}$

(h) $\begin{pmatrix} 2 & 5 & 7 \\ 6 & 3 & 4 \\ 5 & -2 & -3 \end{pmatrix}$

(i) $\begin{pmatrix} 2 & 3 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$

10. Find the inverse of the following $n \times n$-matrix:

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

11. Solve the following matrix equation:

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 0 & 1 & 2 & \dots & n-1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

12. (a) Prove that for any square matrices A and C of equal sizes such that C^{-1} exists $(CAC^{-1})^n = CA^nC^{-1}$.

    (b) Let $f(x)$ be a polynomial. Prove that for any square matrices A and C of equal sizes such that C^{-1} exists $f(CAC^{-1}) = Cf(A)C^{-1}$.

13. Prove that for any 2×2-matrix $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$X^2 - (a+d)X + (ad - bc)I = 0$$

14. [**Extra credit**] Figure out how changes A^{-1} if we

    (a) interchange i-th and j-th rows of A.

    (b) multiply i-th row of A by c.

    (c) add i-th row multiplied by c to the j-th row of A.